STABILITY OF PURE HOMOGENEOUS DEFORMATIONS OF AN ELASTIC PLATE WITH FIXED EDGES

By YI-CHAO CHEN

(Department of Theoretical and Applied Mechanics, Cornell University, Ithaca, New York 14853–1503, USA)

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SUMMARY

An analysis is given of the stability of pure homogeneous deformations of an incompressible elastic plate. The two faces of the plate are free and the displacement of its edges is prescribed. Pointwise conditions for stability are derived by using Fourier transforms and constructing a special displacement field. The conditions obtained are found to be related to a restricted rank-two convexity condition of the strain-energy function. Such a condition is then studied for isotropic materials, resulting in a set of inequalities in terms of the strain-energy function and principal stretches.

1. Introduction

In a recent paper, Chen (1) studied the stability of homogeneous deformations of an incompressible elastic body under dead-load surface tractions. One of the objectives of his paper was to deduce pointwise stability conditions to justify the observation of a biaxial stretch experiment made by Treloar (2) with rubber sheets, in which he reported observing a stable asymmetric deformation under a symmetric load produced by a soft loading device. This suggests, in correspondence with the energy stability criterion, that a properly defined total-energy functional should attain a strict minimum at the asymmetric deformation in a neighbourhood with respect to an appropriate topology. It was shown in (1) that a pure homogeneous deformation, with deformation gradient

\[ F = \sum_{i=1}^{3} \lambda_i e_i \otimes e_i, \quad \lambda_i > 0, \quad \lambda_1\lambda_2\lambda_3 = 1, \]  

(1)

\( \{e_1, e_2, e_3\} \) being an orthonormal basis of \( \mathbb{R}^3 \), of an incompressible isotropic elastic material is a strict relative minimum, defined by using a \( C^1 \) seminorm, of the total energy under a biaxial dead-load surface traction

\[ T = \sum_{\alpha=1}^{2} T_{\alpha} e_{\alpha} \otimes e_{\alpha} \]  

(2)
if and only if the following conditions hold:

\[
\begin{align*}
\frac{\partial W}{\partial \lambda_1} (\lambda_1, \lambda_2) &= T_1, \\
\frac{\partial W}{\partial \lambda_2} (\lambda_1, \lambda_2) &= T_2, \\
T_1 &> 0, \\
T_2 &> 0, \\
T_1 - T_2 &> 0 \quad \text{if } \lambda_1 \neq \lambda_2, \\
\lambda_1 - \lambda_2 &> 0, \\
\lambda_1 - \lambda_3 &> 0, \\
\lambda_2 - \lambda_3 &> 0,
\end{align*}
\]

where \( W = W(\lambda_1, \lambda_2) \) is the reduced strain-energy function of the material. Basically, the conditions (3) are the local strict convexity conditions of the strain-energy function for an incompressible isotropic elastic material. The inequality (3) \( \text{is} \) apparently not satisfied by the stable asymmetric deformation observed in Treloar's experiment for which \( \lambda_1 \neq \lambda_2 \) and \( T_1 = T_2 \). In other words, the asymmetric deformation is \textit{not} stable under the symmetric biaxial dead load.

A possible explanation for this apparent disagreement between theory and experiment could be that the dead-load surface traction is not a perfect model to describe Treloar's experiment. As an ideal soft loading device imposes no kinematical constraints on the boundary, a biaxial dead-load surface traction (2) with \( T_1 = T_2 \) has no orientation preference in the \( \{e_1, e_2\} \)-plane, and therefore should give rise to a deformation or to a class of deformations which also exhibits no orientation preference in this plane. However, this obviously is not the case in Treloar's experiment.

Due to the imperfect machinery in laboratories, it is very difficult to find an accurate and yet manageable model to formulate the boundary conditions in Treloar's experiment. An alternative approach to this problem is, instead of trying to deduce necessary and sufficient conditions for stability from an accurate model, to try to deduce necessary conditions and sufficient conditions for stability by employing idealized models under which the deformation is either more stable or more unstable than that under the real loading device.

Among the sets of kinematically admissible deformations that correspond to various types of boundary conditions, the one for an ideal soft loading device allows for a greater variety of deformations than others. Thus, the corresponding stability conditions (3) are expected to be more restrictive than those corresponding to other loading devices and can serve as a

sufficient condition for stability of the deformations in Treloar's experiment. On the other side, one possible necessary condition for stability is furnished by the well-known Hadamard condition, derived for an ideal hard loading device which specifies the deformation of the body on the entire boundary and therefore corresponds to the smallest possible set of kinematically admissible deformations.

For biaxial stretches of a thin plate, there is room for finding necessary conditions for stability that are stronger than the Hadamard condition. In modelling such an experiment, although we do not know precisely what boundary conditions should be imposed on the edges of the plate, we do know that the two faces of the plate are free. Therefore, the boundary conditions which specify the deformation on the edges and leave the faces free will result in a necessary condition for stability, which, as we shall see, is stronger than the Hadamard condition.

It is the issue of finding stability conditions under this mixed boundary condition that we address in this paper. Such stability conditions have interest of their own as related to the problem of finding additional pointwise conditions for stability when the Hadamard condition is assumed to hold. In general, these stability conditions may depend on geometry, as is suggested by various experiments. The present investigation furnishes an initial result to this issue for a particular case.

A two-dimensional version of this problem has been studied by Shield (3). He obtained a set of stability conditions by considering the stability of a membrane with clamped edges and composed of an isotropic elastic material.

In a related problem, Simpson and Spector (4) studied the problem of finding pointwise conditions for the second variation to be uniformly positive, given that it is strictly positive. The method developed in their analysis could be a useful technique for the further investigation of problems of this type as well as for bifurcation problems in elasticity.

In section 2, we formulate a constrained minimization problem with mixed boundary conditions. The first- and second-variation conditions are stated.

A pointwise condition which is sufficient for the second variation to be positive is deduced in section 3 by using the Fourier transform. Essentially, the condition obtained is a \textit{restricted rank-two convexity condition} of the strain-energy function. Furthermore, a pointwise condition which is weaker than the restricted rank-two convexity condition is found to be necessary for the second variation to be positive. Unlike the restricted rank-two convexity condition, this latter condition depends on the geometry, basically the thickness of the plate, but approaches the restricted rank-two convexity condition as the thickness tends to zero.

In the concluding section 4, the restricted rank-two convexity condition is studied for isotropic materials. A set of inequalities in terms of the reduced

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strain-energy function $W$ and the principal stretches $\lambda_1$ and $\lambda_2$ is derived. This set of inequalities is found to be weaker, as we expect, than (3)$_{308}$ with (3)$_{5}$ and (3)$_{8}$ being replaced by weaker conditions.

2. Basic formulæ

Let $\Omega$ be a bounded regular domain in $\mathbb{R}^2$, and let $h$ be a positive number. We consider an incompressible homogeneous elastic body which, in a fixed reference configuration, occupies the three-dimensional plate-like domain

$$\Omega = S \times (-h, h).$$

A representative particle of the body is denoted by $X \in \Omega$.

The body is isochorically deformed to a configuration $\bar{x}(\Omega)$ by a loading device which specifies the position of the edges of the body and applies zero traction on each of its two faces. Thus, the boundary $\partial \Omega$ of the body consists of two non-empty disjoint parts

$$\partial_1 \Omega = \partial S \times (-h, h) \quad \text{and} \quad \partial_2 \Omega = \partial \Omega - \partial_1 \Omega$$
on which the deformation and the zero traction are prescribed, respectively. In this work, we consider the class of $C^1$ deformations. The set of the kinematically admissible deformations is then given by

$$\mathcal{A} = \{ x \in C^1(\bar{\Omega}; \mathbb{R}^3) : \det \nabla x = 1 \text{ in } \Omega, \ x = \bar{x} \text{ on } \partial_1 \Omega \},$$

where $\bar{x} \in C^1(\bar{\Omega}; \mathbb{R}^3)$ is a given function which satisfies the incompressibility constraint

$$\det \nabla \bar{x} = 1,$$

$\nabla$ denoting the gradient operator.

By the energy stability criterion, the deformation $\bar{x}$ is stable (respectively neutrally stable) if it is a strict (respectively non-strict) minimum of the total energy

$$E(x) = \int_{\Omega} W(\nabla x) \, dX$$
in a subset of $\mathcal{A}$, where we have used the same notation $W$ to denote the strain-energy function of an incompressible anisotropic material, which is assumed to be of class $C^2$ and defined on the set $\mathcal{U}$ of all unimodular tensors. In this work we shall only consider relative minima defined by using a $C^1$ norm. Precisely, the deformation $\bar{x}$ is a relative minimum of $E$ if there is $\varepsilon > 0$ such that

$$E[\bar{x}] \leq E[x]$$
for all $x \in \mathcal{A}$ with $\sup_{\Omega} (|x - \bar{x}|^2 + |\nabla x - \nabla \bar{x}|^2)^{\frac{1}{2}} < \varepsilon$. One of the common approaches to this minimization problem has been to deduce pointwise conditions from the first- and second-variation conditions. A rigorous derivation of these conditions for a mixed boundary-value problem with an incompressibility constraint was recently given by Fosdick and MacSithigh (5). As a particular case (zero traction on $\partial_2 \Omega$ and $\mathcal{A} \subset C^1(\bar{\Omega}; \mathbb{R}^3)$) of their results we have the following.

**Theorem 1.** Suppose that $\bar{x}$ is a relative minimum of $E$ in a $C^1$ neighbourhood of $\bar{x}$ in $\mathcal{A}$. Then there exists a $C^1$ function $p : \bar{\Omega} \rightarrow \mathbb{R}^1$ such that

$$\begin{align*}
\text{Div} \, \bar{W}_F &= \bar{F}^{-T} \nabla p \quad \text{in } \Omega, \\
(\bar{W}_F - p \bar{F}^{-T}) \mathbf{N} &= 0 \quad \text{on } \partial_2 \Omega,
\end{align*}$$

and that

$$\int_{\Omega} \left( \nabla u \cdot \bar{W}_{FF} \nabla u + p \text{ tr } (\bar{F}^{-1} \nabla u)^2 \right) \, dX \geq 0$$

for all $u \in C^1(\bar{\Omega}; \mathbb{R}^3)$ satisfying

$$\begin{align*}
\bar{F}^{-T} \nabla u &= 0 \quad \text{in } \Omega, \\
\mathbf{u} &= 0 \quad \text{on } \partial_1 \Omega,
\end{align*}$$

where

$$\bar{F} = \nabla \bar{x}, \quad \bar{F}^{-T} = (\bar{F}^{-1})^T,$$

$$\bar{W}_F = \frac{\partial W}{\partial \bar{F}} (\nabla \bar{x}), \quad \bar{W}_{FF} = \frac{\partial^2 W}{\partial \bar{F}^2} (\nabla \bar{x}),$$

$\mathbf{N}$ is the unit outward normal to $\partial \Omega$, and $\text{Div}$ denotes the divergence operator.

Here and henceforth, the strain-energy function $W$ is smoothly extended to a neighbourhood of $\mathcal{U}$ in such a way that the derivatives of $W$ up to the second order are well defined and continuous.

Equation (5)$_1$ is the Euler–Lagrange equation for the total energy $E$ subject to the constraint (4). The Lagrange multiplier $p$ corresponds to a hydrostatic pressure required by the incompressibility constraint.

It can also be shown that if the equalities (5) as well as the strict inequality (6) hold for all non-zero $u$ satisfying (7), then $\bar{x}$ is a strict relative minimum of $E$ in the sense that

$$E[\bar{x}] < E[x]$$
for all $x$ in a $C^1$ neighbourhood of $\bar{x}$ in $\mathcal{A}$, with the equality holding only if $x = \bar{x}$.

It is well known (see, for instance, (6, §68)) that if $\partial_2 \Omega = \emptyset$, and if
\( W_{FF} = \text{const. in } \Omega \), a necessary and sufficient condition for (6) to hold is the Hadamard condition. It was shown by Young (7) that the Hadamard condition takes the following form for incompressible elastic materials:

\[
\lambda (a \otimes b) W_{FF}[a \otimes b] \geq 0 \quad \text{for all } a, b \in \mathbb{R}^3 \quad \text{with } a \cdot F^{-T} b = 0. \tag{8}
\]

Strict inequality in (8) is also referred to as the strong ellipticity condition for the elasticity tensor \( W_{FF} \). The Hadamard condition, which is a consequence of the rank-one convexity condition of the strain-energy function, is obviously independent of the geometry of the body. It seems likely that when \( \partial \Omega \neq \emptyset \) a pointwise condition which is necessary and sufficient for (6) to hold for all \( u \) satisfying (7) would depend on the geometry of the body. In general, this problem still remains open.

In the next section, we shall prove some pointwise conditions which are necessary or sufficient for (6) to hold in the particular case where the deformation \( \bar{x} \) is homogeneous.

3. Pointwise stability conditions

The deformation \( \bar{x} \) is said to be homogeneous if \( \nabla \bar{x} \) is constant in \( \Omega \). If such a deformation is a minimum of \( E \), the equation (5), requires that the hydrostatic pressure \( p \) is also constant in \( \Omega \). In this work, we consider a pure homogeneous deformation \( \bar{x} \) with the deformation gradient of the form (1), where the orthonormal basis \( \{ e_1, e_2, e_3 \} \) is assumed to be such that \( e_3 \) is perpendicular to \( \partial \Omega \). This means that the deformation \( \bar{x} \) is a combination of stretches and compressions in or normal to the plane of the plate.

We define a fourth-order tensor \( A \) which has the following component form:

\[
A_{ijkl} = \frac{\partial^2 W}{\partial F_{ij} \partial F_{kl}} + p F^{-1}_{ii} F^{-1}_{jk},
\]

where all components are calculated with respect to the orthonormal basis \( \{ e_1, e_2, e_3 \} \). Then the second-variation condition (6) can be written as

\[
\int_{\Omega} \nabla u \cdot A [\nabla u] \, dX = 0. \tag{9}
\]

In the following proposition we derive, by using an argument of Van Hove (8), a sufficient condition for (9) to hold under (7), which will be shown, in a further proposition, to be also the limit of a necessary condition as \( h \to 0 \).

Let

\[
\Lambda = \{ H \in \text{Lin} : H = a \otimes \tau + b \otimes e_3, \ a, \tau, b \in \mathbb{R}^3, \ \tau \cdot e_3 = 0, \ |\tau| = 1, \quad H \cdot F^{-T} = 0, \ |H| = 1 \}, \tag{10}
\]

where Lin is the set of all linear transformations (tensors) on \( \mathbb{R}^3 \).

\textbf{Proposition 1. If}

\[
H \cdot A[H] = 0 \quad \text{for any } H \in \Lambda, \tag{11}
\]

\textbf{then the inequality (9) holds for any } u \in C^4(\hat{\Omega} ; \mathbb{R}^3) \text{ satisfying (7).}

\textbf{Proof.} By the symmetry of } A, \textbf{(9) holds for any } u \in C^4(\hat{\Omega} ; \mathbb{R}^3) \text{ satisfying (7) if and only if the inequality}

\[
\int_{\Omega} \nabla v \cdot A [\nabla v] \, dX \geq 0 \tag{12}
\]

\textbf{holds for any } v \in C^4(\hat{\Omega} ; \mathbb{C}^3) \text{ satisfying}

\[
g^{-1} \nabla v = 0 \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial \Omega. \tag{13}
\]

Here \( g \) denotes the complex conjugate of \( g \). Now it suffices to show that (11) implies (12). Given \( v \in C^4(\hat{\Omega} ; \mathbb{C}^3) \) satisfying (13), we extend \( v \) to \( \mathbb{R}^2 \times (-h, h) \) by setting \( v = 0 \) on \( \mathbb{R}^2 \times (-h, h) \). Since the extended function is piecewise \( C^1 \) with support lying in \( \Omega \), the functions \( v \) and \( g \) belong to \( L^1 \cap L^2 \) over \( \mathbb{R}^2 \times (-h, h) \). We define their Fourier transforms in \( \mathbb{R}^2 \) as

\[
(\xi(t_1, t_2, X_3)) = \frac{1}{2\pi} \int_{\mathbb{R}^2} v(x_1, x_2, X_3) \exp[-i(t_1 x_1 + t_2 x_2)] \, dx_1 \, dx_2,
\]

\[
(\nabla v)^*(t_1, t_2, X_3) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \nabla v(x_1, x_2, X_3) \exp[-i(t_1 x_1 + t_2 x_2)] \, dx_1 \, dx_2,
\]

\[
(t_1, t_2, X_3) \in \mathbb{R}^2 \times (-h, h), \quad i^2 = -1,
\]

where the components of \( X \) are calculated with respect to \( \{ e_1, e_2, e_3 \} \). Integrating (14a) by parts and using (13), we find that

\[
(\nabla v)^*(t_1, t_2, X_3) = i \hat{v}(t_1, t_2, X_3) \otimes (t_1 e_1 + t_2 e_2) + \frac{\partial \hat{v}}{\partial X_3}(t_1, t_2, X_3) \otimes e_3. \tag{14b}
\]

It follows from (13) that the inner product of (14b) and \( g^{-1} \) vanishes. In particular,

\[
H_1 \cdot g^{-1} = 0, \quad H_2 = 0, \tag{15}
\]

where

\[
H_1 = \left[ \text{Re } \hat{v} \otimes (t_1 e_1 + t_2 e_2) + \text{Im } \frac{\partial \hat{v}}{\partial X_3} \otimes e_3 \right],
\]

\[
H_2 = -\text{Im } \hat{v} \otimes (t_1 e_1 + t_2 e_2) + \text{Re } \frac{\partial \hat{v}}{\partial X_3} \otimes e_3.
\]
By (14a) and Parseval’s theorem, we have
\[
\int_{\Omega} \bar{\nabla} v \cdot A[\nabla v] \, dX = \int_{\mathbb{R}^2 \times (-h, h)} (\bar{\nabla} v)^* \cdot A[(\bar{\nabla} v)^*] \, dt_1 \, dt_2 \, dX_3
\]
\[
= \int_{\mathbb{R}^2 \times (-h, h)} \{H_1 \cdot A[H_1] + H_2 \cdot A[H_2]\} \, dt_1 \, dt_2 \, dX_3.
\]
The inequality (11), with the aid of (10) and (15), ensures that this final integrand is non-negative everywhere. The inequality (12) then follows.

If in the definition (10) of the $\Lambda$, the vectors $\tau$ and $e_k$ were taken arbitrarily, the inequality (11) would be associated with the rank-two convexity condition of the strain-energy function.† Hence, the inequality (11) as it stands will be referred to as a restricted rank-two convexity condition.

The proof of Proposition 1 can be carried out for strict inequality (9) and (11), with non-zero $u \in C^1(\Omega; \mathbb{R}^3)$ satisfying (7).

In the next proposition, we show that a pointwise condition that is weaker than (11) is necessary for (9) to hold under (7). This condition depends on $h$ and approaches (11) as $h \to 0$. The proof of Proposition 2 is modelled on the work of Fichera (see (9)).

**Proposition 2.** Suppose that the inequality (9) holds for all $u \in C^1(\Omega; \mathbb{R}^3)$ satisfying (7). Then there exist constants $C_k > 0$, $k = 1, 2, 3, 4$, depending only on $\bar{F}$, $A$, and $S$, such that
\[
|H| \cdot A[H] \geq - |A| \sum_{k=1}^4 C_k h^k \quad \text{for any } H \in \Lambda. \tag{16}
\]

**Proof.** It suffices to show that (12) implies (16). Let an $H \in \Lambda$ be given with $a, \tau$ and $b$ as specified in (10). We choose a $C^2$ function $\varphi : S \to \mathbb{R}^1$ with non-empty support contained in the interior of $S$, and consider a function $v : \Omega \to \mathbb{R}^3$ of the following form:
\[
v(X) = \bar{F}(\bar{a} \otimes e_3 \varphi(X) e_3 + \text{curl} \{ \varphi(X) (X_3 \bar{a} + 3 \text{i} h^{-1} X_3^3 \bar{b}) \times e_3\}), \tag{17}
\]
where
\[
\bar{a} = \bar{F}^{-1} a, \quad \bar{b} = \bar{F}^{-1} b,
\]
and $\varphi : \Omega \to \mathbb{C}$ is given by
\[
\varphi(X_1, X_2, X_3) = \varphi(X_1, X_2) \exp(\text{i} h^{-1} \tau \cdot X).
\]
A calculation, using (10), shows that the function $v$ defined by (17) satisfies
\[
\nabla v = \bar{F}(\bar{a} \otimes \nabla \varphi - \bar{a} \cdot \nabla \varphi e_3 \otimes e_3 + h^{-1} X_3 \varphi(b \otimes e_3 \otimes e_3 - b \otimes \tau)
\]
\[- X_3 e_3 \otimes \nabla \varphi + \frac{1}{2} h^{-1} X_3^2(b \otimes e_3 \otimes \nabla \varphi + \nabla \varphi e_3 \otimes \tau) +
\]+ ih^{-1} \{\varphi(\bar{a} \otimes \tau + b \otimes e_3) + \frac{1}{2} h^{-1} X_3^2 \bar{F}(\bar{b} \otimes e_3 \otimes \tau +
\]+ X_3(b \otimes \nabla \varphi - \bar{a} \cdot \nabla \varphi e_3 \otimes e_3 - \bar{a} \cdot \nabla \varphi e_3 \otimes \tau)
\]
\[- \frac{1}{2} X_3^2 \bar{F}(\bar{b} \otimes e_3 \otimes \nabla \varphi) \} \exp(\text{i} h^{-1} \tau \cdot X). \tag{18}
\]
Substituting (18) into (12) and carrying out the integration in $X_3$ over $(-h, h)$, we obtain
\[
\int_{\Omega} \bar{\nabla} v \cdot A[\nabla v] \, dX = 2(a \otimes \tau + b \otimes e_3) \cdot A[a \otimes \tau + b \otimes e_3] \times
\]
\[
\int_{S} \varphi^2(X_1, X_2) dX_1 dX_2 + \sum_{k=1}^4 p_k h^k = 0,
\]
where $p_k \in \mathbb{R}^1$, $k = 1, 2, 3, 4$ depend on $A, \bar{F}, H$ and $\varphi$; for example,
\[
p_1 = 2 \int_S \{(a \otimes \nabla \varphi - \bar{a} \cdot \nabla \varphi \bar{F} e_3 \otimes e_3) \cdot A[a \otimes \nabla \varphi - \bar{a} \cdot \nabla \varphi \bar{F} e_3 \otimes e_3] +
\]
\[
+ \frac{1}{2} \varphi^2 b \cdot (a \otimes \tau + b \otimes e_3 \cdot A[\bar{F} e_3 \otimes \tau] +
\]
\[
+ \frac{1}{2} \varphi^2 (b \cdot (b \otimes e_3 \otimes e_3 - b \otimes \tau)) \cdot A[b \cdot (b \otimes e_3 \otimes e_3 - b \otimes \tau)] \} dX_1 dX_2.
\]
It follows from (10) that $|a| \leq 1$ and $|b| \leq 1$. Thus $p_1$ can be bounded by a constant depending only on $A, \bar{F}$ and $\varphi$ as
\[
|p_1| < |A| (1 + |\bar{F}| (|\bar{F}^{-1}|))^2 \int_S (\varphi^2 + 2 |\nabla \varphi|^2) dX_1 dX_2.
\]
Similar results hold for $p_2, p_3$ and $p_4$; this completes the proof.

The technique used in the proofs of Propositions 1 and 2 can also be used to prove similar results for incompressible materials; the proof of Proposition 1 will be almost unchanged, but that of Proposition 2 will have a simpler form than the present incompressible version. The particular form of $v$ in (17) was chosen to satisfy the incompressibility constraint and be such that after integration the leading term of the tensor product of $\nabla v$ was the desired rank-two tensor form. For displacement boundary problems, that is, $\partial_\varnothing \Omega = \varnothing$, a proof similar to that of Proposition 2, which resulted in the Hadamard condition, has been given in (9) for compressible materials and in (5) for incompressible materials.

Comparing (16) with (11) shows that, for a given material and a given $S$, (16) → (11) as $h \to 0$. If (11) is violated for a given material at a given pure homogeneous deformation, then (16) will be violated when $h$ is sufficiently small, that is, the given deformation becomes unstable for a sufficiently thin plate.
Compared to (8), the condition (11) is seen to be stronger than the Hadamard condition, as expected from the fact that releasing part of the boundary may result in a richer class of variations and therefore lead to a stronger condition for a minimum. Yet (11) is, for the same reason, weaker than the condition

$$H \cdot \Lambda[H] \geq 0 \quad \text{for any } H \in \text{Lin with } H \cdot \hat{F}^{-T} = 0,$$

which is necessary and sufficient for (9) to hold under (7) when $\partial \Omega = \emptyset$ (see, for instance, (1, 5, 10)). Fixing part of the boundary may make the class of variations smaller and therefore weaken the condition for a function to be a minimum.

Although we have not been able to deduce a pointwise condition which is necessary and sufficient for (9) to hold under (7), it seems reasonable to expect that such a condition would depend on $S$ and $h$. Spector (11) has proved that for compressible elastic materials the rank-two convexity condition of the strain-energy function is not necessary for the second variation to be non-negative when $\partial \Omega \neq \emptyset$.

The rest of this paper will be devoted to calculating the implications of the restricted rank-two convexity condition (11) for incompressible isotropic elastic materials. We expect that, under the boundary condition $(4)_2$, the inequality $(3)_5$ is no longer necessary for stability, as suggested by Treloar's experiment.

4. Incompressible isotropic elastic materials

An elastic material is isotropic if the strain-energy function $W$ satisfies

$$W(FQ) = W(F)$$

for all proper orthogonal $Q \in \text{Lin}$. It is well known that for an incompressible isotropic material that satisfies the principle of material frame-indifference, the strain-energy function $W$ depends on the deformation gradient $F$ only through two of the principal stretches $\lambda_1$, $\lambda_2$ and $\lambda_3$ of $F$, which are the eigenvalues of $(F^T F)^{1/2}$, for example,

$$W(F) = W(\lambda_1, \lambda_2), \quad (\lambda_1, \lambda_2) \in \mathbb{R}^2_+.$$

Here we have used $W$ to denote two different functions. As the function $W(F)$ is smoothly extended to a neighbourhood of $H$, the function $W(\lambda_1, \lambda_2)$ can be extended to $\mathbb{R}^2_+$ in such a way that the extended function $\hat{W}$ is of class $C^2$ and

$$W(\lambda_1, \lambda_2) = \hat{W}(\lambda_1, \lambda_2, \frac{1}{\lambda_1 \lambda_2}). \quad (19)$$

As was shown in (1), the tensor $A$ can be expressed in terms of $\hat{W}$ as

$$A = \sum_{i,j=1,2,3} \gamma_{ij} e_i \otimes e_i \otimes e_j \otimes e_j + \sum_{i,j=1,2,3} [\sigma_{ij} e_i \otimes e_i \otimes e_j \otimes e_j + \frac{1}{2} \mu_{ij} (e_i \otimes e_j + e_j \otimes e_i) \otimes (e_i \otimes e_j + e_j \otimes e_i)], \quad (20)$$

where

$$\gamma_{ij} = \hat{W}_{ij} = \frac{\partial^2 \hat{W}}{\partial \lambda_i \partial \lambda_j} \left( \lambda_1, \lambda_2, \frac{1}{\lambda_1 \lambda_2} \right),$$

$$\sigma_{ij} = -\frac{\lambda_1 \hat{W}_{ij}}{\lambda_1 \lambda_2},$$

$$\mu_{ij} = \frac{\hat{W}_i - \hat{W}_j}{\lambda_i - \lambda_j} = \frac{\hat{W}_i \lambda_j - \hat{W}_j \lambda_i}{\lambda_i - \lambda_j},$$

$$\hat{W}_i = \frac{\partial \hat{W}}{\partial \lambda_i} \left( \lambda_1, \lambda_2, \frac{1}{\lambda_1 \lambda_2} \right),$$

$$e_{ij} = \begin{cases} 1 & \text{if } (i, j) = (1, 2) \text{ or } (2, 3) \text{ or } (3, 1), \\
0 & \text{otherwise,} \end{cases}$$

$\mu_{ij}$ being calculated in the sense of passing to the limit, for example,

$$\frac{\hat{W}_i - \hat{W}_j}{\lambda_i - \lambda_j} \bigg|_{\lambda_i = \lambda_j} = \lim_{\lambda_i \to \lambda_j} \frac{\hat{W}_i - \hat{W}_j}{\lambda_i - \lambda_j} = \hat{W}_i - \hat{W}_j.$$

We now prove a proposition by a method similar to that used in (12) for a related problem.

**Proposition 3.** The inequality (11) holds for an incompressible isotropic elastic material if and only if the following inequalities hold:

$$\mu_{12} + \mu_{21} \geq 0, \quad (22)$$

$$\left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) \cdot M_1 \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) \geq 0, \quad \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) \cdot M_2 \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) \geq 0 \quad \text{for any } (x_1, x_2) \in \mathbb{R}^2, \quad (23)$$

$$\left( \begin{array}{c} y_1 \\ y_2 \end{array} \right) \cdot M_3(\text{sgn}(y_1, y_2)) \left( \begin{array}{c} y_1 \\ y_2 \end{array} \right) \geq 0 \quad \text{for any } (y_1, y_2, y_3) \in \mathbb{R}^3 \text{ satisfying }$$

$$\lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 = 0, \quad (24)$$

where $M_1$ and $M_2$ are $2 \times 2$ matrices defined by

$$M_1 = \left( \begin{array}{cc} \mu_{12} + \mu_{21} & \mu_{23} - \mu_{32} \\ \mu_{23} - \mu_{32} & \mu_{12} + \mu_{21} \end{array} \right), \quad M_2 = \left( \begin{array}{cc} \mu_{13} + \mu_{31} & \mu_{31} - \mu_{13} \\ \mu_{13} - \mu_{31} & \mu_{13} + \mu_{31} \end{array} \right), \quad (25)$$

$\dagger$ Here we are considering a particular case where all diagonal components, with respect to $(e_1, e_2, e_3)$, of the deformation gradient are positive and where the traction in the $e_3$-direction is zero.
\( M_3(\delta) \) is a 3 \times 3 matrix defined by

\[
M_3(\delta) = \\
\begin{pmatrix}
\gamma_{11} & \gamma_{12} + \sigma_{12} + \frac{\mu_{12} - \mu_{21}}{2} + \frac{\mu_{12} + \mu_{21}}{2} \gamma_{13} + \sigma_{13} \\
\gamma_{21} + \sigma_{21} + \frac{\mu_{12} - \mu_{21}}{2} + \frac{\mu_{12} + \mu_{21}}{2} \gamma_{22} + \gamma_{23} + \sigma_{23} \\
\gamma_{31} + \sigma_{31} & \gamma_{32} + \sigma_{32} & \gamma_{33}
\end{pmatrix}
\]

(26)

\textbf{Proof.} To show sufficiency, let \( H = a \otimes \tau + b \otimes e_3 \) satisfying (10) be given. The condition \( H \cdot F^{-T} = 0 \) requires that

\[
\lambda_1^{-1}a_1r_1 + \lambda_2^{-1}a_2r_2 + \lambda_3^{-1}b_3 = 0,
\]

(27)

where the components of \( a, \tau \) and \( b \) are again calculated with respect to \( \{e_1, e_2, e_3\} \). It follows from (20), (25) and (26) that

\[
H \cdot A[H] = \frac{\mu_{12} + \mu_{21}}{2} \left( a_1r_1 - \delta a_2r_1 \right)^2 + \frac{1}{2} \left( a_3 \tau_2 \right)^2 . M_1 \left( \begin{pmatrix} a_3 \tau_2 \\ b_2 \end{pmatrix} \right) + \\
\frac{1}{2} \left( a_3 \tau_1 \right)^2 . M_2 \left( \begin{pmatrix} a_3 \tau_1 \\ a_2 \tau_2 \end{pmatrix} \right) + \frac{1}{2} \left( a_3 \tau_1 \right)^2 . M_3 \left( \begin{pmatrix} a_3 \tau_1 \\ b_3 \end{pmatrix} \right)
\]

(28)

which, with the aid of equation (27), gives the desired result on setting \( \delta = \text{sgn}(a_1r_1, a_2r_2) \).

We now prove necessity. The inequality (22) follows from setting

\[
a = (1, 0, 0), \quad \tau = (0, 1, 0), \quad b = 0
\]

in (11) with the left-hand side being written as (28). To prove (23), let a non-zero \((x_1, x_2) \in \mathbb{R}^2\) be given and set

\[
a = \frac{1}{(x_1^2 + x_2^2)^{\frac{1}{2}}} (0, 0, x_1), \quad \tau = (0, 1, 0), \quad b = \frac{1}{(x_1^2 + x_2^2)^{\frac{1}{2}}} (0, x_2, 0)
\]

and

\[
a = \frac{1}{(x_1^2 + x_2^2)^{\frac{1}{2}}} (0, 0, x_1), \quad \tau = (1, 0, 0), \quad b = \frac{1}{(x_1^2 + x_2^2)^{\frac{1}{2}}} (x_2, 0, 0),
\]

successively in (11). Finally, for the proof of (24), let a non-zero \((y_1, y_2, y_3) \in \mathbb{R}^3\) with \(\lambda_1^{-1}y_1 + \lambda_2^{-1}y_2 + \lambda_3^{-1}y_3 = 0\) be given. Setting

\[
a = \frac{1}{(y_1^2 + y_2^2)^{\frac{1}{2}}} (\text{sgn} y_1 |y_1|, \text{sgn} y_2 |y_2|, 0),
\]

\[
\tau = \frac{1}{(y_1^2 + y_2^2)^{\frac{1}{2}}} (|y_1|, |y_2|, 0),
\]

\[
b = \frac{1}{(y_1^2 + y_2^2)^{\frac{1}{2}}} (0, 0, y_3), \quad \delta = \text{sgn}(y_1, y_2)
\]

in (11) with the left-hand side being written as (28) gives the desired result.

In a similar way, we can prove Proposition 3 with (11), (22), (23), and (24) being replaced by strict inequalities for non-zero \((x_1, x_2)\) and \((y_1, y_2, y_3)\).

For further results, we prove the following.

\textbf{Lemma 1.} Given real numbers \(A_{11}, A_{12}, A_{22}, B\), the inequality

\[
A_{11}y_1^2 + 2A_{12}y_1y_2 + A_{22}y_2^2 + 2B |y_1, y_2| \geq 0
\]

(29)

holds for any \((y_1, y_2) \in \mathbb{R}^2\) if and only if

\[
\begin{aligned}
A_{11} & \geq 0, \\
A_{22} & \geq 0, \\
(A_{11}A_{22})^{\frac{1}{2}} - |A_{12}| + B & \geq 0. \\
\end{aligned}
\]

(30)

\textbf{Proof.} To show necessity, note that the inequalities (30)\(_{1,2}\) follow from setting \((y_1, y_2) = (1, 0)\) and \((y_1, y_2) = (0, 1)\) successively in (29). If \(A_{11} = 0\), the inequality (29) implies that

\[
\left( \frac{\text{sgn} y_2}{y_1} \right) A_{12} + B + \frac{1}{2} A_{22} \left| \frac{y_2}{y_1} \right| \geq 0 \quad \text{for any } y_1 \neq 0, y_2 \neq 0,
\]

which leads to (30)\(_3\). A similar argument applies when \(A_{22} = 0\). Now if \(A_{11} > 0\) and \(A_{22} > 0\), the inequality (30)\(_3\) follows from setting

\[
(y_1, y_2) = (A_{11}^{\frac{1}{2}} A_{22}^{\frac{1}{2}}, -(\text{sgn} A_{12}) A_{11}^{\frac{1}{2}} A_{22}^{\frac{1}{2}})
\]

in (29).

The sufficiency conclusion is obvious on writing

\[
A_{11}y_1^2 + 2A_{12}y_1y_2 + A_{22}y_2^2 + 2B |y_1, y_2|
\]

\[
\geq A_{11}y_1^2 + 2(B - |A_{12}|) |y_1, y_2| + A_{22}y_2^2
\]

\[
= (A_{11}^{\frac{1}{2}} |y_1| - A_{22}^{\frac{1}{2}} |y_2|)^2 + 2((A_{11}A_{22})^{\frac{1}{2}} - |A_{12}| + B) |y_1, y_2|.
\]

\textbf{Corollary 1.} The inequalities (23) and (24) hold if and only if

\[
\mu_{23} + \mu_{32} \geq 0, \quad \mu_{23} \mu_{32} \geq 0, \quad \mu_{31} + \mu_{13} \geq 0, \quad \mu_{31} \mu_{13} \geq 0;
\]

(31)
also the inequalities (30) hold with

\[
\begin{align*}
A_{11} &= \gamma_{11} - \frac{2\lambda_3}{\lambda_1} (\gamma_{31} + \sigma_{31}) + \frac{\lambda_3^2}{\lambda_1^2} \gamma_{33}, \\
A_{12} &= \gamma_{12} + \sigma_{12} + \frac{\mu_{22} - \mu_{21}}{2} - \frac{\lambda_3}{\lambda_1} (\gamma_{32} + \sigma_{32}) - \frac{\lambda_3}{\lambda_2} (\gamma_{31} + \sigma_{31}) + \frac{\lambda_3^2}{\lambda_1 \lambda_2} \gamma_{33}, \\
A_{22} &= \gamma_{22} - \frac{2\lambda_3}{\lambda_2} (\gamma_{23} + \sigma_{23}) + \frac{\lambda_3^2}{\lambda_2^2} \gamma_{33}, \\
B &= \frac{\mu_{22} + \mu_{21}}{2}.
\end{align*}
\]

(32)

Proof. That (23) and (31) are equivalent is elementary with the aid of (25). By solving \(\lambda_1^2 y_1 + \lambda_2^2 y_2 + \lambda_3^2 y_3 = 0\) for \(y_2\) and substituting it into the inequality (24), we can write (24) as (29) together with (32). The conclusion then follows from Lemma 1.

Lemma 2. The quantities appearing in (22), (31) and (32) can be expressed in terms of \(W(\lambda_1, \lambda_2)\) as

\[
\begin{align*}
\mu_{12} + \mu_{21} &= 2 \frac{\lambda_1 W_1 - \lambda_2 W_2}{\lambda_1^2 - \lambda_2^2}, \\
\mu_{23} + \mu_{32} &= 2 \frac{\lambda_2 W_2}{\lambda_2^2 - \lambda_3^2}, \\
\mu_{23} \mu_{32} &= \frac{W_2^2}{\lambda_2^2 - \lambda_3^2}, \\
\mu_{31} + \mu_{13} &= 2 \frac{\lambda_1 W_1}{\lambda_1^2 - \lambda_3^2}, \\
\mu_{31} \mu_{13} &= \frac{W_1^2}{\lambda_1^2 - \lambda_3^2}, \\
A_{11} &= W_{11}, \\
A_{12} &= W_{12} + \frac{\lambda_2 W_1 - \lambda_1 W_2}{\lambda_1^2 - \lambda_2^2}, \\
A_{22} &= W_{22}, \\
B &= \frac{\lambda_1 W_1 - \lambda_2 W_2}{\lambda_1^2 - \lambda_2^2},
\end{align*}
\]

(33)

where

\[
W_\alpha = \frac{\partial W}{\partial \lambda_\alpha}(\lambda_1, \lambda_2), \quad W_{\alpha \beta} = \frac{\partial^2 W}{\partial \lambda_\alpha \partial \lambda_\beta}(\lambda_1, \lambda_2), \quad \alpha, \beta = 1, 2,
\]

and each expression is calculated in the sense of passing to the limit; for example,

\[
\begin{align*}
(\mu_{12} + \mu_{21})|_{\lambda_1 = \lambda_2} &= 2 \lim_{\lambda_1 \to \lambda_2} \frac{\lambda_1 W_1 - \lambda_2 W_2}{\lambda_1^2 - \lambda_2^2} = W_{11} - W_{12} + \frac{W_1}{\lambda_1}, \\
(\mu_{23} + \mu_{32})|_{\lambda_2 = \lambda_3} &= 2 \lim_{\lambda_2 \to \lambda_3} \frac{\lambda_2 W_2}{\lambda_2^2 - \lambda_3^2} = \frac{1}{2} W_{22}.
\end{align*}
\]

Proof. Differentiating (19) gives

\[
\begin{align*}
W_\alpha &= \dot{W}_\alpha - \frac{\lambda_3}{\lambda_\alpha} \ddot{W}_3, \\
W_{\alpha \beta} &= \dot{W}_{\alpha \beta} - \frac{\lambda_3}{\lambda_\alpha} \ddot{W}_3 + \frac{\lambda_3^2}{\lambda_\alpha^2} \dddot{W}_3 + \frac{2\lambda_3}{\lambda_\alpha} \dddot{W}_3, \quad \alpha, \beta = 1, 2, 3,
\end{align*}
\]

(34)

where use has been made of the condition \(\lambda_1 \lambda_2 \lambda_3 = 1\). The desired result then follows from substitution of (34) into (21) and (32).

Combining Proposition 3, Corollary 1 and Lemma 2, we arrive at the following.

COROLLARY 2. For an incompressible isotropic elastic material, the inequality (11) holds if and only if the following inequalities hold:

\[
\begin{align*}
\frac{\lambda_1 W_1 - \lambda_2 W_2}{\lambda_1 - \lambda_2} &\leq 0, \\
\frac{W_2}{\lambda_2 - \lambda_3} &\geq 0, \\
W_{33} &\geq 0, \\
W_1 &\geq 0, \\
W_{11} &\geq 0, \\
W_{22} &\geq 0,
\end{align*}
\]

(35)

where all expressions are again calculated in the sense of passing to the limit.

The inequalities (35) are seen to be weaker than the conditions (3) as we noted in section 1. In a similar way, we can prove the following proposition concerning the strict inequality (11).

PROPOSITION 4. For an incompressible isotropic elastic material, the strict inequality (11) is equivalent to the following inequalities:

\[
\begin{align*}
\frac{\lambda_1 W_1 - \lambda_2 W_2}{\lambda_1 - \lambda_2} &> 0, \\
\lambda_2 - \lambda_3 &> 0, \\
W_2 &> 0, \\
\lambda_1 - \lambda_3 &> 0, \\
W_1 &> 0, \\
W_{11} &> 0, \\
W_{22} &> 0,
\end{align*}
\]

(36)

Shield (3) derived the inequalities (36) by considering the stability of an isotropic elastic membrane uniformly stretched with clamped edges. The inequalities (36), essentially required by the stability under shears in the planes perpendicular to the traction-free surface of the plate, are not predictable by the membrane theory that Shield used.
As was shown by the strict-inequality versions of Theorem 1 and Proposition 1, strict inequality in (11) is sufficient for the deformation $\bar{x}$ to be a strict relative minimum of the total energy $E$ under the boundary condition (4). Thus, the inequalities (36) can serve as a sufficient condition for stability of a pure homogeneous deformation of a plate-like body composed of an incompressible isotropic elastic material with fixed edges. Comparing (36) with (3), we see that the condition $(3)_1$ is replaced by the weaker condition $(3)_2$, which, as expected, is satisfied by the stable asymmetric deformation under a symmetric load observed in Treloar's experiment.

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